WORKED EXAMPLES

Yr11 Specialist Mathematics

Proof

Everyone is unique, so should be every lesson.

YUMING LIN
WHY TUTORING
info@why-tutoring.com

5. Prove that the sum of squares of two odd integers cannot be a perfect square.

Plan:

1) Odd integers can be expressed in the form $2k + 1$.

2) We would start with $(2a + 1)^2 + (2b + 1)^2$, and we would somehow show that it cannot be written as a perfect square.

3) We can first simplify the expression above, and then we determine whether we need direct proof or proof by contradiction.

Solution:

Let the two integers be $2a + 1$ and $2b + 1$, where $a, b \in \mathbb{Z}$. Then we have

$$\begin{aligned}(2a + 1)^2 + (2b + 1)^2 &= (4a^2 + 4a + 1) + (4b^2 + 4b + 1)\\ &= 4(a^2 + b^2 + a + b) + 2\\ &= 4c + 2\end{aligned}$$

where $c = a^2 + b^2 + a + b \in \mathbb{Z}$

Next, we would prove that a number in the form $4c + 2$ cannot be a perfect square.

Let a perfect square number be n^2, where $n \in \mathbb{Z}$. Then we have the following two cases.

Case 1: If n is odd, we let $n = 2k + 1$, where $k \in \mathbb{Z}$, then

$$\begin{aligned}n^2 &= (2k + 1)^2\\ &= 4k^2 + 4k + 1\\ &= 4(k^2 + k) + 1\\ &= 4c + 1\end{aligned}$$

where $c = k^2 + k \in \mathbb{Z}$.

Case 2: If n is even, we let $n = 2k$, where $k \in \mathbb{Z}$, then

$$\begin{aligned}n^2 &= (2k)^2\\ &= 4k^2\\ &= 4c\end{aligned}$$

where $c = k^2 \in \mathbb{Z}$.

Therefore, a perfect square number is either in the form $4c + 1$ or $4c$. This indicates that the sum of squares of two odd integers, which is in the form $4c + 2$, cannot be a perfect square.

Extension:

Sometimes we need to prove another theorem or identity in our solution, to get what we are asked to show, i.e., a proof in a proof. Being familiar with some commonly used theorems would provide more ideas to our planning.

Some popular ones are as follows.

- If $a^2 + b^2 = c^2$, then a, b and c cannot all be odd integers.
- A perfect square number is either in the form $4k + 1$ or $4k$, where $k \in \mathbb{Z}$.
- If n^2 is divisible by a prime number, then n itself is also divisible by that prime number.
- $|x| \geq x$, $|x| + |y| \geq |x + y|$, $|x| - |y| \leq |x - y|$
- $x^2 + y^2 \geq 2xy$, $x + y \geq 2\sqrt{xy}$, $(x + y)^2 \geq 4xy$

You can try proving the above theorems as practice.

12. Let k be a positive integer. Prove that if $2^{k+2} + 3^{3k}$ is divisible by 5, then $2^{k+3} + 3^{3k+3}$ is also divisible by 5.

Plan:

1) We would somehow derive that $2^{k+2} + 3^{3k} = 5b$ given that $2^{(k+2)} + 3^{3k} = 5a$.

2) Such a question requires us to express $2^{k+3} + 3^{3k+3}$ in terms of $2^{k+2} + 3^{3k}$ to reach the conclusion.

3) As the exponents of the two expressions are similar, we may need to apply index laws to split and rewrite $2^{k+3} + 3^{3k+3}$.

Solution:

We first try to express $2^{k+3} + 3^{3k+3}$ in terms of $2^{k+2} + 3^{3k}$ using index laws.

$$\begin{aligned} 2^{k+3} + 3^{3k+3} &= 2^{(k+2)+1} + 3^{3k+3} \\ &= 2^{(k+2)} \cdot 2 + 3^{3k} \cdot 3^3 \\ &= 2^{(k+2)} \cdot 2 + 3^{3k} \cdot 27 \\ &= 2^{(k+2)} \cdot 2 + 3^{3k} \cdot (2 + 25) \\ &= 2^{(k+2)} \cdot 2 + 3^{3k} \cdot 2 + 3^{3k} \cdot 25 \\ &= (2^{(k+2)} + 3^{3k}) \cdot 2 + 3^{3k} \cdot 25 \end{aligned}$$

As $2^{(k+2)} + 3^{3k}$ is divisible by 5, we can let $2^{(k+2)} + 3^{3k} = 5a$, where $a \in \mathbb{Z}$. Then the expression can be further simplified.

$$\begin{aligned} 2^{k+3} + 3^{3k+3} &= (2^{(k+2)} + 3^{3k}) \cdot 2 + 3^{3k} \cdot 25 \\ &= (5a) \cdot 2 + 3^{3k} \cdot 25 \\ &= 10a + 3^{3k} \cdot 25 \\ &= 5(2a + 3^{3k} \cdot 5) \\ &= 5b \end{aligned}$$

where $b = 2a + 3^{3k} \cdot 5 \in \mathbb{Z}$.

Therefore, if $2^{k+2} + 3^{3k}$ is divisible by 5, then $2^{k+3} + 3^{3k+3}$ is also divisible by 5.

Extension:

This is a proof question closely related to algebra. Such a question does not require us to understand abstract concepts, but we need to be able to use the correct algebraic technique to express the conclusion in terms of the hypothesis.

15. Prove that when the sum of squares of two distinct positive integers is doubled, the result can still be written as the sum of two distinct square numbers.

Plan:

1) We first interpret question. It wants us to prove that $2(m^2+n^2)=a^2+b^2$, where $m,n \in \mathbb{Z}^+, a,b \in \mathbb{Z}$, and $m \neq n, a \neq b$.

2) We can try some examples and see how this works.
$$2(1^2+2^2)=10=1^2+3^2$$
$$2(1^2+3^2)=20=2^2+4^2$$
$$2(3^2+4^2)=50=1^2+7^2$$
$$2(2^2+5^2)=58=3^2+7^2$$
Did you recognize any pattern about the values of m,n,a,b?

3) We want to rewrite $2(m^2+n^2)$ into the sum of two perfect squares. Maybe the perfect square formula $a^2+2ab+b^2=(a+b)^2$ is helpful.

Solution:

Let $m,n \in \mathbb{Z}^+$, and $m \neq n$. Then

$$\begin{aligned}
2(m^2+n^2) &= (m^2+n^2)+(m^2+n^2)\\
&= (m^2+n^2)+(m^2+n^2)+2mn-2mn\\
&= (m^2+2mn+n^2)+(m^2-2mn+n^2)\\
&= (m+n)^2+(m-n)^2\\
&= a^2+b^2
\end{aligned}$$

where $a=m+n, b=m-n$.

As $m,n \in \mathbb{Z}^+$, then $a,b \in \mathbb{Z}$. Also, as $m \neq n$, and $m \neq 0, n \neq 0$, then $m+n \neq m-n$, that is $a \neq b$.

Therefore, when the sum of squares of two distinct positive integers is doubled, the result can still be written as the sum of two distinct square numbers.

Extension:

This is a challenging question. Although the algebra is not complex, it is very hard to think of introducing $+2mn-2mn$ terms in the expression to complete the square. This is why writing out some examples may help us see the pattern. Experience in algebra and some 'Eurika' moment is also needed in this one.