WORKED EXAMPLES

Yr11 Specialist Mathematics

Vectors

Everyone is unique, so should be every lesson.

YUMING LIN

WHY TUTORING

info@why-tutoring.com

1. **Consider the vectors $\vec{p} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\vec{q}$ being of magnitude 2 in the positive x direction. Neatly and accurately draw the following on a separate piece of graph paper.**

   a) $2\vec{p} + 3\vec{q}$

   b) $\vec{p} - \vec{q}$

   c) $\hat{\vec{p}}$

   d) The projection of $\vec{p}$ on $\vec{q}$

Plan:

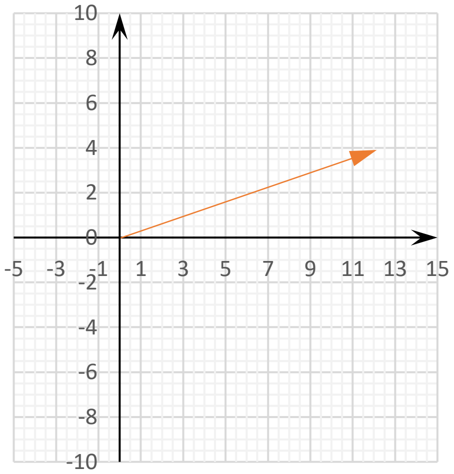
1) This is a basic algebraic question about vectors. We need to be clear about the concepts of drawing vectors, the sum/difference of vectors, the unit vectors and the projection.

2) We should first find $\vec{q}$ and then answer the questions.

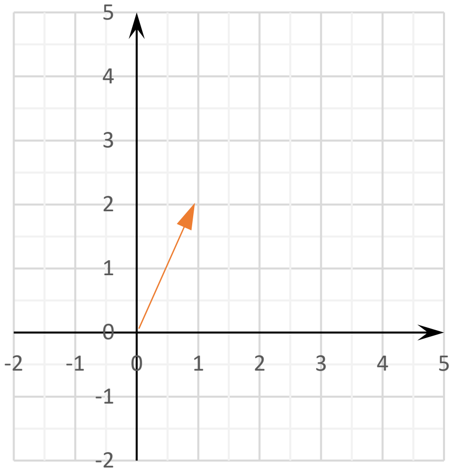
Solution:

According to the question, $\vec{q} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

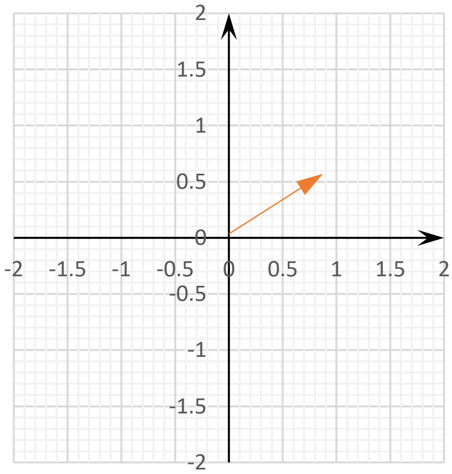
a) $2\vec{p} + 3\vec{q} = 2\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$. The graph is as follows.



b) $\vec{p} - \vec{q} = \begin{bmatrix}3\\2\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$. The graph is as follows.



c) $\hat{\vec{p}} = \frac{\vec{p}}{|\vec{p}|} = \frac{\begin{bmatrix}3\\2\end{bmatrix}}{\sqrt{3^2+2^2}} = \frac{\begin{bmatrix}3\\2\end{bmatrix}}{\sqrt{13}} = \begin{bmatrix}\frac{3}{\sqrt{13}}\\\frac{2}{\sqrt{13}}\end{bmatrix}$. The graph is as follows.



d) To find the projection of $\vec{p}$ on $\vec{q}$, we first find $\hat{\vec{q}}$.

$$\hat{\vec{q}} = \frac{\vec{q}}{|\vec{q}|} = \frac{\begin{bmatrix}2\\0\end{bmatrix}}{\sqrt{2^2+0^2}} = \frac{\begin{bmatrix}2\\0\end{bmatrix}}{2} = \begin{bmatrix}1\\0\end{bmatrix}$$

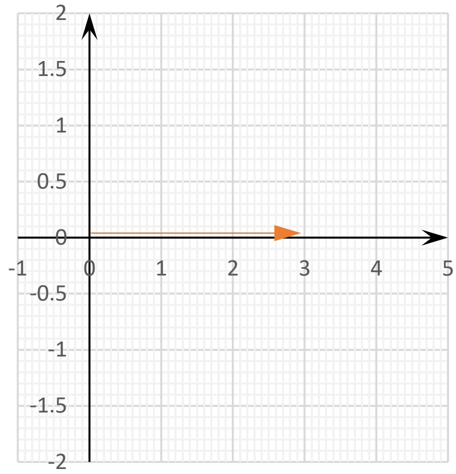
Then we find the scalar resolute.

$$\vec{p} \cdot \hat{\vec{q}} = \begin{bmatrix}3\\2\end{bmatrix} \cdot \begin{bmatrix}1\\0\end{bmatrix} = 3 \times 1 + 2 \times 0 = 3$$

Thus, the vector resolution of $\vec{p}$ on $\vec{q}$ is

$$(\vec{p} \cdot \hat{\vec{q}})\hat{\vec{q}} = 3\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$$

The graph is as follows.



Extension:

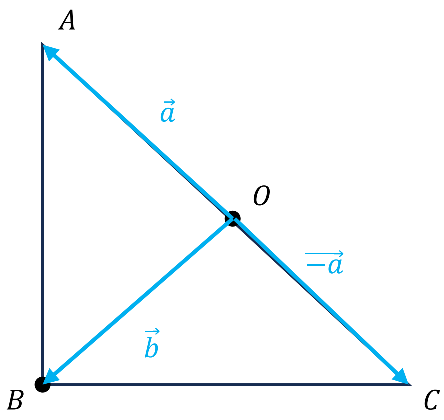
This question is not hard or tricky, but it requires solid understanding of basic vector concepts and sketching skills. You need to be familiar with the process of finding any linear combination of vectors, unit vectors, scalar resolute and vector resolute of vectors. Make sure your can get the answer correctly and efficiently.

5. Show that the midpoint of the hypotenuse of a right-angled triangle is equidistant to the three vertices of the triangle.

Plan:

1) We would make use of the perpendicular legs of the triangle to construct an equation.

2) We can let the two legs be $\vec{a}$ and $\vec{b}$, or we can let the midpoint of the hypotenuse be the origin to assign $\vec{a}$ and $\vec{b}$ to the edges from the midpoint.



Solution:

As shown in the diagram above, ΔABC is a right-angled triangle where $\angle ABC = 90°$. O is the midpoint of AC. We let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, and thus $\overrightarrow{OC} = -\vec{a}$.

We have

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\vec{a} + \vec{b} = \vec{b} - \vec{a}$$

$$\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$$

As $AB \perp CB$, then $\overrightarrow{AB} \cdot \overrightarrow{CB} = 0$, that is

$$(\vec{b} - \vec{a}) \cdot (\vec{b} + \vec{a}) = 0$$

$$\left(\vec{b}\right)^2 - (\vec{a})^2 = 0$$

$$\left|\vec{b}\right|^2 - |\vec{a}|^2 = 0$$

$$\left|\vec{b}\right|^2 = |\vec{a}|^2$$

$$\left|\vec{b}\right| = |\vec{a}|$$

Also, as $|\vec{a}| = |-\vec{a}|$, then we have $\left|\vec{b}\right| = |\vec{a}| = |-\vec{a}|$, that is, $OB = OA = OC$.

Therefore, the midpoint of the hypotenuse of a right-angled triangle is equidistant to the three vertices of the triangle.

Extension:

Assigning $\vec{a}$ and $\vec{b}$ to the sides in the most convenient way would significantly simplify our algebra. It is also important to notice that $\overrightarrow{OC} = -\vec{a}$. You can compare the work above with the case if we let $\overrightarrow{BA} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$, and see which solution is easier.

11. Prove that the two diagonals of a rhombus bisect each other at right angles.

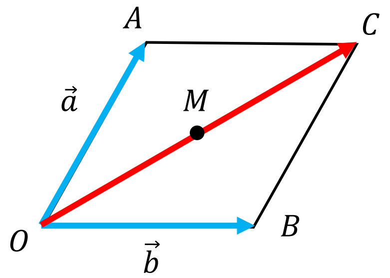
Plan:

1) When dealing with a parallelogram/rhombus/rectangle, we construct a diagram first and let the two adjacent sides be $\vec{a}$ and $\vec{b}$. Then we express the segments related in the proof in terms of $\vec{a}$ and $\vec{b}$.

2) To prove the diagonals bisect each other, we need to prove that the midpoint on one diagonal lies on the other diagonal, which requires the proof of collinearity.

3) To prove the diagonals are perpendicular, we need to show that the dot product of the diagonals is zero.

Solution:



We construct a rhombus $OACB$, and let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, thus $|\vec{a}| = |\vec{b}|$. We also let M be the midpoint of OC, as shown in the figure on the right.

Then the two diagonals are

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\vec{a} + \vec{b}$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + \overrightarrow{OB} = \vec{a} + \vec{b}$$

and thus

$$\overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM} = \overrightarrow{AO} + \frac{1}{2}\overrightarrow{OC} = -\vec{a} + \frac{1}{2}(\vec{a} + \vec{b}) = \frac{1}{2}(-\vec{a} + \vec{b})$$

As $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$, then M lies on AB and is also the midpoint of AB, that is AB and OC bisect each other.

Also we have

$$\overrightarrow{AB} \cdot \overrightarrow{OC} = (-\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (\vec{b})^2 - (\vec{a})^2 = |\vec{b}|^2 - |\vec{a}|^2 = 0$$

Thus AB and OC are perpendicular to each other.

Therefore, the two diagonals of a rhombus bisect each other at right angles.

Extension:

In a geometric proof question, we need to be familiar with the techniques in proving various types of relations, including bisecting, perpendicular, parallel, colinear, etc. Also, we need to know how to prove the converse or the inverse if asked. For this question, the converse is also true. Try to prove the converse yourself: 'If two diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus'.

16. Points P and Q have position vectors $\vec{p}$ and $\vec{q}$, with reference to an origin O, and M is the point on PQ such that

$$\beta\overrightarrow{PM} = \alpha\overrightarrow{MQ}$$

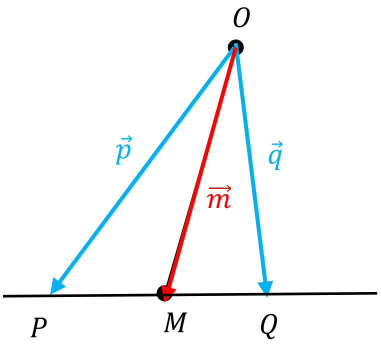
a) Prove that the position vector of M is given by $m = \frac{\beta\vec{p}+\alpha\vec{q}}{\alpha+\beta}$.

Write the position vectors of P and Q as $\vec{p} = k\vec{a}$ and $\vec{q} = l\vec{b}$, where k and l are positive real numbers and $\vec{a}$ and $\vec{b}$ are unit vectors.

b) Prove that the position vector of any point on the internal bisector of $\angle POQ$ has the form $\lambda(\vec{a}+\vec{b})$.

c) If M is the point where the internal bisector of $\angle POQ$ meets PQ, show that

$$\frac{\alpha}{\beta} = \frac{k}{l}$$



Plan:

1) Part a) can be solved using vector addition.

2) Part b) and c) requires more work in dot product and angle bisectors. We need to recall that the diagonal of a rhombus bisects the internal angle.

Solution:

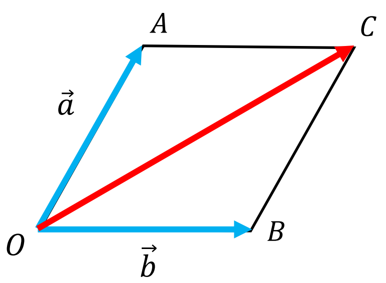
a) As shown in the diagram above, $\beta\overrightarrow{PM} = \alpha\overrightarrow{MQ}$ indicates $\frac{PM}{MQ} = \frac{\alpha}{\beta}$ and $\frac{PM}{PQ} = \frac{\alpha}{\alpha+\beta}$. Thus, we have

$$\overrightarrow{PM} = \frac{\alpha}{\alpha+\beta}\overrightarrow{PQ} = \frac{\alpha}{\alpha+\beta}\left(\overrightarrow{PO}+\overrightarrow{OQ}\right) = \frac{\alpha}{\alpha+\beta}(-\vec{p}+\vec{q})$$

and

$$\vec{m} = \overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM} = \vec{p} + \frac{\alpha}{\alpha+\beta}(-\vec{p}+\vec{q}) = \frac{\beta\vec{p}+\alpha\vec{q}}{\alpha+\beta}$$

b) As shown in the diagram on the right, $\vec{a}$ and $\vec{b}$ are unit vectors, and $OACB$ is a rhombus. Thus, the diagonal OC would be a valid bisector of $\angle AOB$, as the diagonal of a rhombus bisects the internal angle.



As $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = \vec{a} + \vec{b}$, then any vector which is a scalar multiple of $\overrightarrow{OC}$ is also a valid bisector of $\angle AOB$, which is equal to $\angle POQ$.
That is, the position vector of any point on the internal bisector of $\angle POQ$ has the form $\lambda(\vec{a} + \vec{b})$, where $\lambda \in \mathbb{R}\backslash\{0\}$.

c) If M lies on the internal bisector of $\angle POQ$, according to part b), we have

$$\vec{m} = \lambda(\vec{a} + \vec{b})$$

where $\lambda \in \mathbb{R}\backslash\{0\}$.
Also, from part a), we have

$$\vec{m} = \frac{\beta\vec{p} + \alpha\vec{q}}{\alpha + \beta} = \frac{\beta(k\vec{a}) + \alpha\left(l\vec{b}\right)}{\alpha + \beta} = \frac{\beta k\vec{a} + \alpha l\vec{b}}{\alpha + \beta}$$

Equating the two equations above yields

$$\lambda\left(\vec{a} + \vec{b}\right) = \frac{\beta k\vec{a} + \alpha l\vec{b}}{\alpha + \beta}$$

$$\lambda(\alpha + \beta)\vec{a} + \lambda(\alpha + \beta)\vec{b} = \beta k\vec{a} + \alpha l\vec{b}$$

As $\vec{a}$ and $\vec{b}$ are linearly independent, then we have

$$\begin{cases} \lambda(\alpha + \beta) = \beta k \\ \lambda(\alpha + \beta) = \alpha l \end{cases}$$

and thus

$$\beta k = \alpha l$$
$$\frac{\alpha}{\beta} = \frac{k}{l}$$

Extension:

Did you notice that part a) is about the ratio division formula we learned in coordinate geometry? Were you aware of that part b) is like the work we did in Q10? Did you recall the properties of the linear independent vectors we used in part c)? What is the significance of the result we have shown in part c)? How is it related to our proof in Q19?