WORKED EXAMPLES

Yr11 Specialist Mathematics

Combinatorics

Everyone is unique, so should be every lesson.

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1. Show that $P_r^n - P_r^{n-1} = rP_{r-1}^{n-1}$.

Plan:

1) This is a pure algebraic proof with permutation. We need to simplify both sides into factorial notation.

2) Make well use of identities such as $\frac{n!}{(n-1)!} = n$, $\frac{n!}{(n-2)!} = n(n-1)$, etc. Also bear in mind that $m!$ is always a factor of $n!$ if $m < n$.

Solution:

$$
\begin{aligned}
LHS &= P_r^n - P_r^{n-1} \\
&= \frac{n!}{(n-r)!} - \frac{(n-1)!}{(n-1-r)!} \\
&= \frac{n!}{(n-r)!} - \frac{(n-1)!}{(n-1-r)!} \times \frac{(n-r)}{(n-r)} \\
&= \frac{n!}{(n-r)!} - \frac{(n-1)!\,(n-r)}{(n-r)!} \\
&= \frac{n! - (n-1)!\,(n-r)}{(n-r)!} \\
&= \frac{n(n-1)! - (n-1)!\,(n-r)}{(n-r)!} \\
&= \frac{(n-1)!\,\big(n-(n-r)\big)}{(n-r)!} \\
&= \frac{(n-1)!\,r}{(n-r)!} \\
RHS &= rP_{r-1}^{n-1} \\
&= \frac{r(n-1)!}{\big((n-1)-(r-1)\big)!} \\
&= \frac{r(n-1)!}{(n-r)!} \\
\therefore LHS &= RHS
\end{aligned}
$$

Extension:

Finding the common denominator of factorial notations is critical in simplification and proof questions like this. For example, the lowest common denominator of $\frac{1}{(n-1)!}$, $\frac{1}{(n-2)!}$, and $\frac{1}{(n+1)!}$ is $(n+1)!$, and the three fractions can therefore be combined into $\frac{(n+1)n+(n+1)n(n-1)+1}{(n+1)!}$.

9. Bob is about to hang his eight shirts in wardrobe. He has four different styles of shirts, two identical ones of each particular style. How many arrangements are possible if no two identical shirts are next to one another?

Plan:

1) This is a question about inclusion-exclusion principle.

2) We would consider the number of the following arrangements.

   Case1: One pair of identical shirts are together, which may include case 2, 3, and 4.
   Case 2: Two pairs of identical shirts are together, which may include case 3 and 4.
   Case 3: Three pairs of identical shirts are together, which may include case 4.
   Case 4: All four pairs of identical shirts are together.

3) Then, we apply the inclusion-exclusion principle to work out the number of arrangements such that at least one pair of identical shirts are together.

4) Lastly, we take it away from the total number of arrangements to get the answer.

Solution:

We define the following events.

A: The first pair of identical shirts are together.

B: The second pair of identical shirts are together.

C: The third pair of identical shirts are together.

D: The fourth pair of identical shirts are together.

Then we would solve for $n(\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D})$, which is equivalent to $n(\overline{A \cup B \cup C \cup D})$ according to De Morgan's Law.

As $n\left(\overline{A \cup B \cup C \cup D}\right) = n(Total) - n(A \cup B \cup C \cup D)$, our task is to work out $n(A \cup B \cup C \cup D)$. This can be done using the inclusion-exclusion principle as follows.

$$
\begin{aligned}
n(A \cup B \cup C \cup D) \\
&= \big(n(A) + n(B) + n(C) + n(D)\big) \\
&- \big(n(A \cap B) + n(A \cap C) + n(A \cap D) + n(B \cap C) + n(B \cap D) \\
&+ n(C \cap D)\big) \\
&+ \big(n(A \cap B \cap C) + n(A \cap C \cap D) + n(A \cap B \cap D) + n(B \cap C \cap D)\big) \\
&- n(A \cap B \cap C \cap D)
\end{aligned}
$$

We first investigate the case when one pair of shirts are together, for example, the event A. We group the pair of shirts and arrange the group with the other six shirts. This would be a permutation of grouped items and identical items, which can be calculated as

$$n(A)=\frac{7!}{(2!)^3}\times\frac{2!}{2!}=630$$

We can tell that $n(B)=n(C)=n(D)=n(A)$, then

$$n(A)+n(B)+n(C)+n(D)=C_1^4\times 630=2520$$

Next, we investigate the case when two pairs of shirts are together, for example, the event $A\cap B$. We group the two pairs of shirts and arrange the two groups with the other four shirts. This would also be a permutation of grouped items and identical items, which can be calculated as

$$n(A\cap B)=\frac{6!}{(2!)^2}\times\left(\frac{2!}{2!}\right)^2=180$$

Also, we can tell that $n(A\cap B)=n(A\cap C)=n(A\cap D)=n(B\cap C)=n(B\cap D)=n(C\cap D)$, then

$$n(A\cap B)+n(A\cap C)+n(A\cap D)+n(B\cap C)+n(B\cap D)+n(C\cap D)$$
$$=C_2^4\times 180=1080$$

Similarly, we can work out the case when three pairs of shirts are together, for example, $n(A\cap B\cap C)$, and then multiply it by C_1^4 to get the number of arrangements in case 3.

$$n(A\cap B\cap C)=\frac{5!}{2!}\times\left(\frac{2!}{2!}\right)^3=60$$

$$n(A\cap B\cap C)+n(A\cap C\cap D)+n(A\cap B\cap D)+n(B\cap C\cap D)=C_1^4\times 60=240$$

Lastly, we work out the case when all four pairs of shirts are together, i.e., $n(A\cap B\cap C\cap D)$.

$$n(A\cap B\cap C\cap D)=4!\times\left(\frac{2!}{2!}\right)^4=24$$

Therefore, we have

$$n(A\cup B\cup C\cup D)=2520-1080+240-24=1656$$

Given that the total number of arrangements without any restrictions is

$$n(Total)=\frac{8!}{(2!)^4}=2520$$

We can work out the final answer as follows.

$$\begin{aligned} n\left(\overline{A \cup B \cup C \cup D}\right) &= n(Total) - n(A \cup B \cup C \cup D) \\ &= 2520 - 1656 \\ &= 864 \end{aligned}$$

Extension:

The inclusion-exclusion principle is often combined with De Morgan's Law, and we need to be clear the principle is to find the union of multiple events. If we are after a intersection, like in this question, we need to apply the De Morgan's Law to express it in terms of a union first.

Also, do not forget the C_r^n we used in the solution, which counts the possible selections of the pairs of shirts which are to be together.

16. Twelve points are arranged in a circle. Three triangles are formed using these points as vertices, such that the vertices of the triangles are distinct. How many ways can this be achieved if the triangles do not overlap?

Plan:

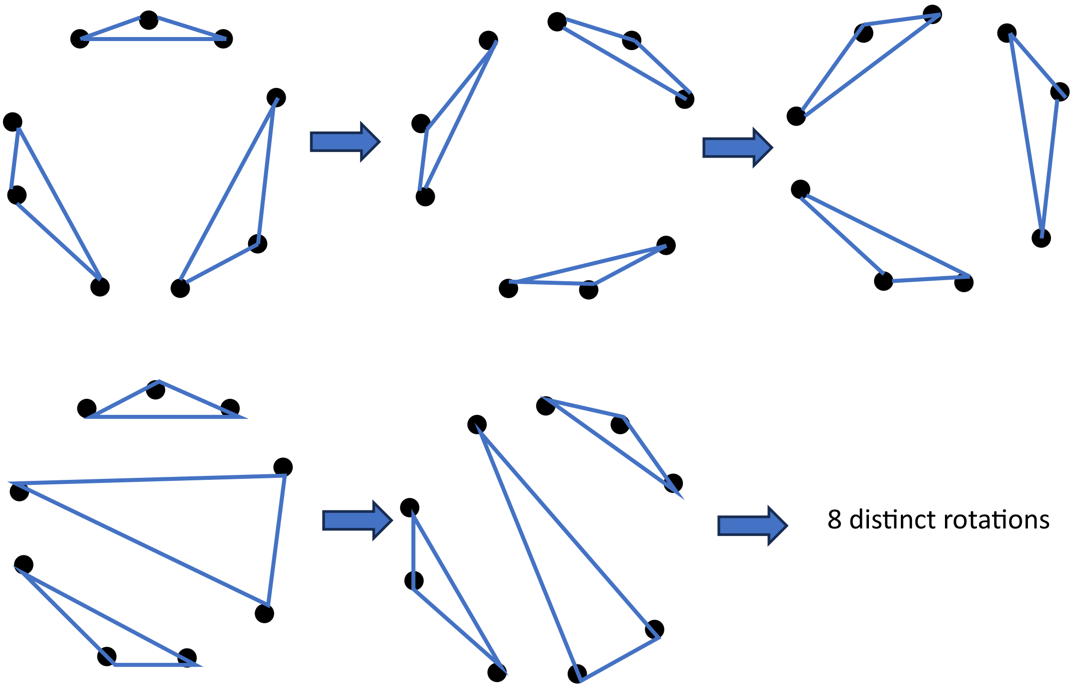
1) How to apply combinatorics in forming triangles from points? Shall we use P_r^n or C_r^n? Why are the 12 points specified to be in a circle? Why the vertices of the triangles must be distinct?

2) How shall we deal with the restriction that the triangles cannot overlap?

Solution:

As the 12 points are arranged in a circle, none of them are collinear. Thus, any three points can form a triangle. Also, as the vertices are distinct, it would require 9 points for three triangles. Therefore, we have C_9^{12} ways of forming three triangles out of the 12 points.

Now we focus on the 9 points selected. To ensure the three triangles do not overlap, we need to sketch the possible scenarios and count the possibilities as follows. We can see that there are only 3+8 possible cases.



Therefore, the total number of ways of achieving this is $C_9^{12} \times 11 = 2420$.

Extension:

Breaking down a complex question into several simple ones is a commonly used technique in practical. Drawing out scenarios in combinatorics may not seems that lengthy and scary in some circumstances as there could be just a few possible cases to exhaust.

17. How many ways can we divide 12 people into five teams, where two teams contain 3 people each, and the remaining three teams contain 2 people each.

Plan:

1) Dividing a group of people into teams requires C_r^n.

2) Be careful with the teams with equal number of people.

3) We first select 3 people from 12 people (C_3^{12}), then select another 3 people from the remaining 9 people (C_3^9), and we keep on going until we have the last team determined.

4) Then we need to consider the identical teams. For example, there are two teams of 3 people, so selecting 3 people for Team A first or for Team B first does not matter. Similarly, the order of determining the three teams of 2 people does not matter either.

Solution:

$$\frac{C_3^{12} \times C_3^9 \times C_2^6 \times C_2^4 \times C_2^2}{2! \times 3!} = 138600$$

Extension:

If the teams are different, then we do not divide the combination by factorials. For example, if 10 basketballers are divided into two teams of five, and one team is the starting lineup while the other is the substitute, then the number of ways of achieving this would just be $C_5^{10} \times C_5^5$.

28. Suppose that every point in the plane is coloured in one of two colours. Prove that no matter how this is done, there must be two points of the same colour exactly 1 unit apart.

Plan:

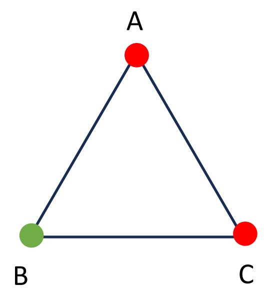
1) From the question we can probably tell it is related to the pigeonhole principles. But how should we apply the formula?

2) It looks like that the pigeons are the points, and the holes are the colours. We do not know what role the '1 unit apart' plays in this question, so we leave that piece of information apart for now.

3) Since there are two colours (two holes), we need three points (three pigeons) to make the pigeonhole principles work. It does require a bit intuition and experience now to see that if we let the three points be three vertices of an equilateral triangle, then two vertices must be of the same colour. Lastly, we let the side length of the triangle be 1 unit so that it satisfies the requirement of the question.

Solution:

We construct an equilateral triangle with side length of 1 unit in the plane, and let the three vertices be A, B, and C, then A, B, and C are 1 unit apart from each other. According to the pigeonhole principle, when the three vertices are coloured in one of two colours, at least two of the points are of the same colour, as shown on the right.



As A, B, and C are points 1 unit apart in the plane, and at least two of them are of the same colour, then we have shown that there exists at least two points on the plane of the same colour which are 1 unit apart.

Extension:

Interpreting a geometric problem using pigeonhole principle requires practice and experience. The critical step is to identify what should be the pigeons and what should be the holes. Two similar questions are given below as your practice.

Show that among any six points in an equilateral triangle with side length of a cm, there are at least two points which are no more than $\frac{a}{2}$ cm apart.

Show that among any five points in a square with side length of a cm, there are at least two points which are no more than $\frac{a}{\sqrt{2}}$ cm apart.